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RESEARCH ARTICLE

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Embedded theoretical quality option pricing in Treasury bond futures—Starting from the definition deviation of conversion factor

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Abstract

Unlike ordinary futures, Treasury bond futures are a kind of complex financial derivatives with multiple Treasury bonds as the underlying, which can be settled on multiple dates. China's Treasury bond futures contract embeds a quality option, rolling timing option, and month end timing option, and these options restrict each other, making the pricing of Treasury bond futures extremely difficult. Quality option plays a dominant role in these three options. This article creatively divides quality options into theoretical quality option caused by the definition deviation of conversion factor and disturbance quality option caused by the market factors except for interest rate. Using the bond valuation method based on the yield to maturity curve, this article puts forward the embedded theoretical quality option and China's Treasury bond futures pricing models. For the empirical test, the dataset covers a 10-year Treasury bond futures contract in 151 working days. The results show that the relative error between our model and the actual closing price of the Treasury bond futures is small compared with the cost of carry model, which excludes any embedded options. This research constructs a practical and straightforward pricing model of embedded theoretical quality option.

KEYWORDS

conversion factor, embedded option, theoretical quality option, timing option, Treasury bond futures

1 | INTRODUCTION

1.1 | Background

Treasury bond futures are an advanced financial future with Treasury bonds as its underlying. The Treasury bond futures contract would be settled by physical delivery at maturity dates. In 1992, China launched the first pilot of the Treasury bond futures market. At that time, the financial market was still an emerging market with an imperfect regulatory mechanism. Some investors engaged in illegal trading for profit, resulting in vicious events, called '319' and '327'. These events directly led to the failure of the first attempts for Treasury bond futures, but its trading was finally stopped in 1995. At that time, Treasury bond future was a contract for only a single specially designated Treasury bond. A single Treasury bond is limited, compared with the Treasury bond futures that can be traded in unlimited quantities. A single Treasury bond is relatively too small and easy for manipulation by investors with a large number of funds. The illegal manipulation events inevitably occur.

After 20 years, in 2013, China's Treasury bond futures market restarted and its 5-year contract was listed and traded due to improving the financial market and the regulatory mechanism, deepening the market-oriented reform of interest rate, and learning from the previous experience and lessons. Subsequently, China launched 10- and 2-year Treasury bond futures in 2015 and 2018, respectively. Under the new rules, single specific Treasury bonds were stopped, but they added quotation trading in the form of nominal standard bonds and took multiple Treasury bonds meeting specific conditions (known as deliverable bonds) which are physically deliverable. For example, the 5-year Treasury bond futures contract allows the party with the short position to deliver any Treasury bond that has a maturity of more than 4 years but less than 5.25 years. When a particular bond is delivered, a parameter known as its conversion factor (CF) defines the invoice price received for the bond. The new specification has essential difference from the original specification in Treasury bond futures in three aspects: ① The deliverable underlying is not unique, and the short party has the right to choose the cheapest-to-deliver bond for delivery. ② The delivery time can last for several days, and the short party can make a delivery declaration on any business day from the first working day to the second Friday in the expiration month. ③ The transaction ends and the settlement price is set at 11:30 AM on the last trading day. After that, the short party waits for the opportunity to select the delivery bond in the spot bond market before 3:15 PM because the Treasury bond market continues working until that time.

These three aspects provide the party with the short position with three rights: ① The right to choose the cheapest-to-deliver (CTD) bond, known as the quality option; ② The right to choose the optimal delivery date, and mentioned as the rolling timing option in this article. This option is similar to the American option that can be exercised at any time; ③ The right to buy cheap bonds, after setting the settlement price, is referred to as the end of month option, and mentioned as month end timing option in this article. The latter two options are both caused by the uncertainty of the delivery time, so they are combined as a timing option. The generation of quality options is mainly caused by varying the CF given by China Financial Futures Exchange.[1] The CF converts the price of the deliverable Treasury bond to the price of the nominal standard bond. Their rational conversion ratio can only be determined on the delivery date. The CF is unreliable if determined before the delivery date, producing quality options. Oviedo (2006) shows the poor performance of the CF system in US CBOT Treasury bond futures trading. Rodolfo also used the real discount rate of future cash flows of delivery bonds with a par value of US $1 to improve the calculation method of the CF. Ramzi and Michèle (2017) analysed the limitations of deliverable bonds in the US Treasury bond futures market. They show that the cheapest deliverable bonds of long-term Treasury bond futures from 1994 to 1999 and from 2015 to 2020 have been fixed on a specific Treasury bond, increasing the risk of a short squeeze. The main reason for these problems is the deviation of the CF's definition, which solidifies the deliverable bond with the shortest duration into the cheapest deliverable bond. Kane and Marcus (1984), Ramzi et al. (2009) and Michèle and Ramzi (2018) also raised this question. There is an international consensus on the irrationality of the CF definition.

1.2 | Research significance

These three options depend on and interact with each other, complicating the price setting of Treasury bond futures. The types and details of embedded options in China's Treasury bond futures market are different from the mature Treasury bond futures market. Take the US Treasury bond futures as an example. Its three options are dissimilar to China, and include a particular option, Wild Card Play.[2] It is inappropriate to directly use the theoretical model of the western mature market for China's Treasury bond futures. A reasonable pricing model helps the Treasury bond futures market improve its price discovery and hedging functions, and promotes the stability of the Treasury bond futures market. Therefore, it is crucial to develop and customize a theoretical model for China's Treasury bond futures market based on the special embedded options and comprehensively consider their impacts on the price of Treasury bond futures.

The existing literature on the pricing of Treasury bond futures is mainly concentrated in the 10 or 20 years after launching the US Treasury bond futures market in the 1980s, while they decreased significantly in recent years. The reason is that after decades of development, the research space of the original direction is relatively small and challenging. The complexity of embedded options in Treasury bond futures makes the relevant theories stagnate in the existing direction, and opening up a new channel is urgent.

Despite an international consensus on the irrationality of the CF system, there is no literature on the pricing of Treasury bond futures that directly starts with the CF to price Treasury bond futures. With regard to this research gap, this article creatively decomposes the embedded quality option of Treasury bond futures into two parts: the theoretical quality option caused by the definition deviation of the CF and the disturbance quality option caused by the deviation of the actual price of deliverable bonds from the theoretical value, to construct the pricing model of Treasury bond futures. After excluding the common factor that affects the Treasury bond futures price—market interest rate, the value deviations of deliverable Treasury bonds are independent of each other. This classification of quality options greatly simplifies the embedded option pricing process of Treasury bond futures. Thus, in the theoretical quality option pricing stage, we can only focus on the theoretical values, while we can only concentrate on the independent value deviations in the subsequent disturbance quality option pricing.

Many researchers have pointed out that the quality option is the most essential option affecting the price of Treasury bond futures, and the disturbance quality option needs to be priced together with the rolling timing option or month end timing option (Gay & Manaster, 1984; Kane & Marcus, 1986b). Therefore, this article only focuses on the impact of the theoretical quality option to investigate the pricing of Treasury bond futures, and makes an empirical analysis based on historical data. This research tries to open up a new research direction and develop a complete pricing model on Treasury bond futures.

2 | LITERATURE REVIEW

Pricing has always been the focus of the research on Treasury bond futures. Various researchers have studied options embedded in Treasury bond futures due to the cost of carry model, which ignores these options. Most of the models are related to quality option pricing since it plays a leading role in all the other options.

2.1 | Cost of carry model

The earliest Treasury bond futures pricing theory ignores the impact of embedded options, among which the cost of carry model is the most representative one. Cornell and French's (1983a) developed the simple and primary model. Then Cornell and French's (1983b) modified this model, based on the complete market hypothesis, to study the stock index futures. The model states that the price of stock index futures is theoretically equal to the spot price plus the holding cost (or cash interest) minus the dividend income. Later, researchers used this model for the pricing of Treasury bond futures. This price equals the forward price of the cheapest deliverable bond at the delivery time divided by the corresponding CF. Yong (2018) studied the impact of interest rate on pricing based on the cost of carry model. The results show that the inter-bank capital interest rate is more representative than the exchange capital interest rate. Wang Su-Sheng and Yong-Rui (2017) analysed the effectiveness of the cost of carry model in the pricing of China's Treasury bond futures and the change rule of the cheapest deliverable bonds. The above cost of carry model focuses on the 'carry' cost, that is, the expenses received or paid because it carries the spot Treasury bonds, and disregards the option value embedded in Treasury bond futures, resulting in a significant difference between the actual and theoretical prices.

2.2 | Theoretical framework of the quality option pricing

The quality option models have three theoretical categories: (1) exchange option pricing method, (2) models which consider the term structure and randomness of interest rate and (3) replication method.

An exchange option is an option to exchange one asset for another. Margrabe (1978) gave an explicit solution similar to Black Scholes for binary convertible options based on the assumption of the geometric Brownian motion of the underlying asset. Stulz (1982) also formulated the option pricing for two deliverable assets under the assumption that the assets obey geometric Brownian motion. Hedge (1988) used the exchange option method and pointed out that the embedded three-month quality option value is about 0.5% of the face value of the Treasury bond futures contract. Hemler (1990) used Margrabe (1978) model to price the quality option of Treasury bond futures. Hedge (1990) compared the advantages and disadvantages of three different exchange option measurement methods on the pricing of quality option. Grieves and Marcus (2005) believed that only two deliverable bonds are related to the price of Treasury bond futures which have the longest or shortest duration, in the case of a flat yield curve. Hence, the exchange option model is appropriate for pricing Treasury bond futures. Grieves et al. (2010) empirically analysed the exchange option model according to Grieves and Marcus (2005), and tested the nature of the negative convexity of Treasury bond futures price implied by this model. Based on the results, this model can accurately price the Treasury bond futures if the yield curve is flat; otherwise, only the numerical method can price the Treasury bond futures. Sun (2014) used the binary exchange option pricing model, developed by Margrabe (1978), to make an empirical analysis on the quality option of Treasury bond futures in China. Zhang (2013) analysed the quality options of Treasury bond futures, and compared three option pricing models. Zeng (2015) used the stock asset exchange option model, formulated by Margrabe

(1978), to price the quality option of Treasury bond futures and modified the future price.

A few studies use the replication method to study quality option. For example, Balbas and Reichardt (2010) used Treasury bond futures and a virtual forward contract to replicate the quality option. The final price of the quality option is determined by the present value of all deliverable bonds and the price of Treasury bond futures in the current market. This research also gives the upper and lower bounds of the quality option determined by the market bid ask spread and dividends.

The prices of deliverable Treasury bonds mainly depend on the changes in market interest rate. Therefore, many studies employ the term structure and randomness of interest rate to study the quality option. Kane and Marcus (1986a) estimated the yield to maturity of Treasury bonds by regressing the cross-sectional data of Treasury bond futures, and assessed its term structure by the Monte Carlo simulation method. Then, they estimated the CTD bonds for delivery according to this term structure, and obtained the embedded options of Treasury bond futures. The results show that the values of the embedded option are range between 1.9% and 6.2%. Carr (1988), for the first time, used the one factor interest rate model to price quality option. Carr and Chen (1996) extended the model developed by Carr (1988) to two factor interest rate model. Ritchken and Sankarasubramanian (1992) used the stochastic process of forward interest rate developed by Heath et al. (1992) to price the quality option of Treasury bond futures. In addition, Nunes and De Oliveira (2007) proposed a corresponding quality option pricing model of Treasury bond futures based on multi factor HJM (Heath-Jarrow-Morton) interest rate process. Rendleman (2004) used the BDT interest rate process to describe the instantaneous risk-free interest rate, and obtained the value of embedded options of Treasury bond futures. Then, this pricing method is integrated into the arbitrage strategy to calculate the optimal hedging ratio. Furthermore, Chen and Ge (2017) proposed an improved double tree spliced BDT (Black-Derman-Toy) interest rate model to price the quality option. Based on the no arbitrage pricing principle in equilibrium, Xiong (2019) used the dynamic interest rate model to describe the interest rate process, and priced the Treasury bond future to predict the trend of the price difference between the Treasury bond futures and the spot, which guides the trading.

2.3 | Comments

The pricing theory of Treasury bond futures defines the cost of carry model in a relatively simple way, but it ignores the embedded options, and the resulted price is the upper bound of the Treasury bond futures price.

The exchange option method is widely applicable to the pricing models of quality option, but its accuracy is insufficient when the conditions are moderately relaxed. Moreover, the exchange option method that gives an analytical solution can be deduced only when the delivery process is subject to a specific assumption. These basic assumptions are applicable to equity assets, but inapplicable to bond assets. The replication method also needs strong assumptions on the bonds processes to replicate the quality option. Hence, the two kinds of models are insufficiently accurate. Considering the term structure and randomness of interest rate explains how the value of Treasury bond futures changes with the interest rate, grasps the most important properties of the underlying assets, and solves the problems of the first two models. This model is also reliable and generic for studying the other embedded options.

Despite strong consensus on the effective role of the definition deviation of CF in quality option, no study considers the pricing of quality option from this perspective. For this reason, it is urgent to open up a new research direction on the pricing of embedded options in Treasury bond futures. Based on the term structure of yield to maturity, this article considers the impact of definition deviation of CF on quality option according to the cost of carry theory, price the quality option, and finally derive the pricing model of Treasury bond futures.

3 | GENERATION MECHANISMS AND PRICING MODEL OF THEORETICAL QUALITY OPTION

3.1 | Generation mechanisms and definition

The Treasury bond prices affect the quality option. The market interest rate plays an essential role in the Treasury bond. In addition, the prices of Treasury bonds are also affected by the supply and demand, the market environment, and some characteristics of Treasury bonds. Our analysis shows that interest rate is the only factor that affects the quality option caused by the definition deviation of the CF. This effect implies that the total quality option also includes the opportunities formed by the impact of market disturbance on Treasury bond prices. Therefore, we divide quality options into two parts: theoretical quality options caused by the fluctuation of market interest rate or by the definition deviation of CF and disturbance quality option caused by market disturbance other than the interest rate.

The emergence of embedded quality options in Treasury bond futures is inseparable from its special delivery rules. When the Treasury bond futures contract is settled, the party with the short position should choose an available bond to deliver, and receives the settlement cash for a bond with￥100 face value delivered from the party with the long position. The following equation calculates settlement cash.

$$\text{Most recent settlement price} \times \text{Conversion factor} + \text{Accrued interest}.$$

The CF is a ratio between the quotation of the bond delivered and the standard bond, making the value of the bond delivered and the delivery fee as close as possible. The China Financial Futures Exchange published the CFs of bond delivered corresponding to different Treasury bond futures contracts before listing the Treasury bond futures contracts. It is equal to the quoted price that the bond would have per Yuan of principal on the first day of the delivery month, based on the assumption that the interest rate for all maturities equals the coupon rate of the standard bond. Currently, the nominal coupon rates are 3% and 6% for standard bonds of all contracts in China and the US, respectively.

$$\begin{aligned}\mathrm{CF} &= \frac{\frac{C}{f}}{\left(1+\frac{r}{f}\right)^{f*\frac{x}{12}}} + \frac{\frac{C}{f}}{\left(1+\frac{r}{f}\right)^{f*\frac{x+\frac{12}{f}}{12}}} + \frac{\frac{C}{f}}{\left(1+\frac{r}{f}\right)^{f*\frac{x+2*\frac{12}{f}}{12}}} + \ldots + \frac{\left(\frac{C}{f}+1\right)}{\left(1+\frac{r}{f}\right)^{f*\frac{x+(n-1)*\frac{12}{f}}{12}}} - \frac{C}{f}\times\frac{12-fx}{12} \\ &= \frac{1}{\left(1+\frac{r}{f}\right)^{\frac{xf}{12}}}\left[\frac{C}{f}+\frac{C}{r}\left(1-\frac{1}{\left(1+\frac{r}{f}\right)^{n-1}}\right)+\frac{1}{\left(1+\frac{r}{f}\right)^{n-1}}\right] - \frac{C}{f}\times\frac{12-fx}{12}.\end{aligned} \tag{1}$$

where CF is conversion factor, $r=3\%$, x indicates the number of months between the maturity month of Treasury bond futures and the next coupon month of the deliverable bond, C is the coupon of the deliverable bond per Yuan of principal, n is the number of remaining cash flows, and f is the coupon frequency.

If the Treasury bond futures contract is settled on the first day of the maturity month and the market interest rate at that time is just equal to the coupon rate of the standard bond (i.e., 3%), this CF can perfectly convert the standard and deliverable bonds to each other. In case of ignoring the changes in the price of Treasury bonds caused by market supply and demand, there is no difference for the party with a short position to choose any deliverable bond for delivery. This matter is because the cost of purchasing a deliverable bond from the market is equal to the settlement cash paid by the party with a long position. However, the delivery date is not unique, and the market interest rate cannot be precisely equal to 3%. In this case, the CF of deliverable and standard bonds is inadequate at delivery time. Therefore, there will be the most favourable situation for the party with a short position to deliver a particular bond, which is the CTD bond. In addition to the uncertainty of the delivery date and the discount rate, the definition of the CF also approximately treats the maturity of the deliverable bond as multiple months (but not days). Although this treatment simplifies the CF calculation, it increases the gap between the predetermined CF and the rational conversion ratio.

From the definition of the CF, we can see that it is a clean price[3] in form, not a ratio. The reason is that the nominal standard bond is issued on the first day of the maturity month, and the coupon rate is 3% in China, consistent with the interest rate used for discount in the definition of the CF. Thus, the clean price of the standard bond with a face value of ￥1 has a value of ￥1 on the first day of the maturity month. Therefore, the original CF formula is similar to clean price, while it is a ratio after dividing the clean price of the standard bond by 1.

Suppose the market interest rate is unequal to 3%, or the delivery date is not necessarily the first day of the delivery month. In that case, the rational CF for the delivery of a deliverable bond i (suppose that there are n deliverable bonds) on a future delivery date j (suppose there are m settlement dates) is the ratio of the theoretical clean price of the deliverable bond ($P^*_{ij}, i=1:n, j=1:m$) and the standard bond ($F^*_j, j=1:m$). Equation (2) shows this case[4].

$$\mathrm{CF}^*_{ij} = \frac{P^*_{ij}}{F^*_j} = \frac{\frac{C^i}{(1+r^{ij})^{t^{ij}_1}} + \frac{C^i}{(1+r^{ij})^{t^{ij}_2}} + \ldots + \frac{C^i}{(1+r^{ij})^{t^{ij}_{n_{ij}}}} - AI_{ij}}{\frac{C^`}{(1+r^j)^{t^j_1}} + \frac{C^`}{(1+r^j)^{t^j_2}} + \ldots + \frac{C^`}{(1+r^j)^{t^j_{n_j}}} - AI_j} \quad i=1:n, j=1:m. \tag{2}$$

where C^i denotes the coupon of ith deliverable bond, C' presents' the coupon of the standard bond, $t^{ij}_1,\ldots t^{ij}_{n_{ij}}$ and $t^j_1,\ldots t^j_{n_j}$ are the times corresponding to cash flows of the ith deliverable and standard bonds at the jth settlement date, r^{ij} and r^{j} are the yield to maturity for discount, and AI_{ij} and AI_j are the accrued interest. We refer CF_{ij}^* as the theoretical CF corresponding to ith deliverable bond and jth settlement date. In this definition, $P_{ij}^* = F_j^* * \mathrm{CF}_{ij}^*$ and $P_{ij}^* + \mathrm{AI}_{ij} = F_j^* * \mathrm{CF}_{ij}^* + \mathrm{AI}_{ij}$ define the settlement cash. The settlement cash for the future is just equal to the value of the deliverable bond. In case of ignoring the difference of the timing values settled on different dates, under these theoretical CFs, there is no difference for the party with a short position to choose which bond to deliver, that is, the theoretical quality option is zero.

As a result, the bond and the date the party with the short position choose to deliver must maximize Equation (3) at the settlement date.

$$\max_{i,j}\left\{F_j^* * \mathrm{CF}_i + \mathrm{AI}_{ij} - \left\{F_j^* * \mathrm{CF}_{ij}^* + \mathrm{AI}_{ij}\right\}\right\} = \max_{i,j} F_j^* * \left(\mathrm{CF}_i - \mathrm{CF}_{ij}^*\right) \tag{3}$$

$$i = 1:n, j = 1:m.$$

where $\mathrm{CF}_i, i = 1:n$ are determined before the list of the futures, and $F_j^*, j = 1:m$ and $\mathrm{CF}_{ij}^*, i = 1:n, j = 1:m$ are obtainable.

The bond, which maximizes Equation (3), is referred to as the CTD bond. This maximization indicates that the CTD bond and the optimal delivery date generate the highest delivery revenue for the party holding a short position based on the current market interest rate term structure at the delivery date. If the embedded options are disregarded, investors will quote the Treasury bond futures as the forward clean price of the current CTD bond divided by the corresponding CF (i.e., the quotation of the cost of carry model), that is $F_j^{\mathrm{CTD}} = \frac{P_{\mathrm{CTD},j}^*}{\mathrm{CF}_{\mathrm{CTD}}}, j = 1:m$. Equation (4) shows the adjusted rational CFs based on the cost of the carry model.

$$\mathrm{CF}_{ij}^{\mathrm{CTD}} = \frac{P_{ij}^*}{F_j^{\mathrm{CTD}}}\ i = 1:n, j = 1:m, \tag{4}$$

where $\mathrm{CF}_{\mathrm{CTD},j}^{\mathrm{CTD}} = \mathrm{CF}_{\mathrm{CTD}}, j = 1:m$.

If the embedded quality option is considered under the quotation of cost of carry model, investors have the option to select an alternative bond for delivery, as determined by Equation (5). This equation defines the theoretical value of the quality option.

$$\mathrm{ThQuOption} = \max_{i,j}\left\{F_j^{\mathrm{CTD}} \times \mathrm{CF}_i + \mathrm{AI}_{ij} - \left\{F_j^{\mathrm{CTD}} \times \mathrm{CF}_{ij}^{\mathrm{CTD}} + \mathrm{AI}_{ij}\right\}\right\} = \max_{i,j} F_j^{\mathrm{CTD}} \times \left(\mathrm{CF}_i - \mathrm{CF}_{ij}^{\mathrm{CTD}}\right), i = 1:n, j = 1:m. \tag{5}$$

where $\{\mathrm{CF}_i, i = 1:n\}$ are given by China Financial Futures Exchange, and $\left\{F_j^{\mathrm{CTD}}, \mathrm{CF}_{ij}^*\ i = 1:n, j = 1:m\right\}$ are all can be calculated as above. Since $F_j^{\mathrm{CTD}} * \left(\mathrm{CF}_{\mathrm{CTD}} - \mathrm{CF}_{\mathrm{CTD},j}^{\mathrm{CTD}}\right) = 0$ for all j, the theoretical quality option is always larger than or equal to 0.

If bond I and delivery date J maximize Equation (5), the rational Treasury bond future price is $F = \frac{P_{IJ}^*}{\mathrm{CF}_I}$, which represents our final estimated price. The quality option value defined by Equation (5) differs significantly from traditional put options, e.g., European put options, $\mathrm{put}_t = e^{-r(T-t)}\widehat{E}(\mathrm{Max}(K - S_T; 0))$ or American put options, $\mathrm{put}_t = \sup_{t \le \tau \le T} \widehat{E}\left[e^{-r(\tau - t)}(K - S_\tau)^+\right]$. These pricing models aim to determine the theoretical values on valuation date, necessitating the discounting of expected payoffs. Conversely, our primary objective here is to determine the rational quotations for Treasury bond futures. Notably, the quality option only impacts the Treasury bond future on delivery dates, and there is no need to discount them.

3.2 | Bond valuation based on the term structure of yield to maturity

According to the definition deviation of the CF discussed in the previous part, the rational CF mainly depends on the clean prices of deliverable bonds and standard bond on the delivery date. The clean prices can be estimated by the instantaneous interest rate pricing model or by discounting all the remaining cash flows from the market interest rate term structure. This research initially used the Vasicek (1977) model to assess the clean prices, but the results significantly deviate from the market price. The main reason is that the instantaneous interest rate model expresses all the term structure of interest rates in the future through a small number of parameters, making it quite different from the actual interest rate term structure in the market. This effect results in apparent differences between the model and the actual results. Thus, this method is inadequate for the pricing of Treasury bond futures. Appendix A outlines the complete process.

Therefore, this research uses the Treasury bond yield curve published by the website of China Central Depository Clearing Co., Ltd. to estimate the future prices of Treasury bonds. According to the compilation instructions published on the official website of China Central Depository Clearing Co., Ltd., our method adopts the Hermite polynomial interpolation model. This model is appropriate for the actual situation of China's bond market for the yield to maturity curve to find the yield at those non-key time points. The specific methods are discussed as follows.

Suppose n key time points, $0 \le t_1 < \ldots < t_i < \ldots < t_n \le T$, and its corresponding yield as $y_1, \ldots, y_i, \ldots, y_n$. Then, Equation (6), which is the cubic Hermite polynomial interpolation model, calculates the yield to maturity of $y(t)$ corresponding to any time point t (suppose $t_i < t < t_{i+1}$).

$$y(t) = y_i H_1 + y_{i+1} H_2 + d_i H_3 + d_{i+1} H_4. \tag{6}$$

In which,

$$H_1 = 3\left(\frac{t_{i+1}-t}{t_{i+1}-t_i}\right)^2 - 2\left(\frac{t_{i+1}-t}{t_{i+1}-t_i}\right)^3;$$

$$H_2 = 3\left(\frac{t-t_i}{t_{i+1}-t_i}\right)^2 - 2\left(\frac{t-t_i}{t_{i+1}-t_i}\right)^3;$$

$$H_3 = \frac{(t_{i+1}-t)^2}{t_{i+1}-t_i} - \frac{(t_{i+1}-t)^3}{(t_{i+1}-t_i)^2};$$

$$H_4 = \frac{(t-t_i)^3}{(t_{i+1}-t_i)^2} - \frac{(t-t_i)^2}{t_{i+1}-t_i};$$

$d_i = y'(t_i)$[5], $d_{i+1} = y'(t_{i+1})$.

This interpolation method calculates the yield at time t where $t_i < t < t_{i+1}$, and uses the three interest rates corresponding to t_{i-1}, t_i, and t_{i+1}.

The Hermite polynomial interpolation model is a non-parametric model, distinct from the well-known Nelson–Siegel–Svensson (NSS) model. The NSS model is a static yield curve model which estimates and predicts the term structure of interest rates. It was firstly developed by Nelson and Siegel (1987), and extended by Svensson (1994). Then, Gürkaynak et al. (2007) associated the NSS model with a six-parameter function to fit US daily Treasury yields from 1961 to 2006, and ultimately derived the yield curve for each respective day.

Equation (7) calculates the value of a t-year Treasury bond on the valuation date according to the bond valuation model published on the official website of China Central Depository Clearing Co., Ltd.

$$\mathrm{PV} = \frac{C/f}{(1+y/f)^w} + \frac{C/f}{(1+y/f)^{w+1}} + \ldots + \frac{\frac{C}{f}}{\left(1+\frac{y}{f}\right)^{w+N-1}} + \frac{FV}{\left(1+\frac{y}{f}\right)^{w+N-1}}, \tag{7}$$

where PV indicates the dirty price of the bond, y is the yield to maturity corresponding to this bond, C is the coupon paid every year, f represents the frequency, N denotes the number of remaining cash flows, FV is the Face value, $w = \frac{\mathrm{Day}}{\mathrm{Day}_0}$ indicates the next coupon time, Day represents the number of days between the valuation date and the next coupon payment date (including the first day but excluding the final day), and Day_0 denotes the number of days between the last coupon date and the next coupon date.

In addition, initially, this research intends to employ the forward yield to maturity as a means to estimate the future values of bonds. However, the forward yield to maturity curve provided on the official website of China Central Depository Clearing Co., Ltd. only involves several specific forward terms, which cannot meet our needs. Therefore, we utilize Equations (8) to calculate the implied forward bond values. Equation (8) is expressed as follows:

$$\mathrm{PV}_1 = (\mathrm{PV}_0 - D)(1 + y\prime/f)^{t_1 * f}. \tag{8}$$

where PV_0 stands for the estimated bond value on the valuation date, derived from Equation (7); PV_1 indicates the forward bond value corresponding to the future delivery date; l denotes the number of remaining cash flows between the valuation date and the forward delivery date; t_1 is the time elapsed between the valuation date and the forward delivery date, calculated by dividing the number of days by 365; y' represents the yield to maturity corresponding time t_1; Furthermore, D signifies the present value of all coupon payments made on the bond during the period between the valuation date and the forward delivery date. It is calculated by summing up the present values of individual coupon payments, each discounted at rate y'. The calculation of D is given by the following formula:

$$D = \frac{C/f}{(1+y'/f)^w} + \frac{C/f}{(1+y'/f)^{w+1}} + \ldots + \frac{C/f}{(1+y'/f)^{w+l-1}}. \tag{9}$$

3.3 | Empirical steps

Based on the definition of theoretical quality option and the valuation method of bond price, our method follows 6 steps (assuming n deliverable bonds and m deliverable dates).

Step 1: Input the Treasury bond yield curve corresponding to each date in the pricing time range;

Step 2: Calculate the theoretical clean prices P_{ij}^*, $i=1:n$, $j=1:m$, corresponding to bond i in date j;
Step 3: Calculate the theoretical clean prices $F_j^*, j=1:m$;
Step 4: Find CTD bond according to $\max_{i,j}\left\{F_j^* * CF_i + AI_{ij} - \left\{P_{ij}^* + AI_{ij}\right\}\right\} = \max_{i,j}\left\{F_j^* * \mathrm{CF}_i - P_{ij}^*\right\}, i=1:n,$ $j=1:m$, and then find the special future price under the cost of carry model $F_j^{\mathrm{CTD}} = \frac{P_{\mathrm{CTD},j}^*}{\mathrm{CF}_{\mathrm{CTD}}}, j=1:m$;

Step 5: Calculate the theoretical CFs corresponding to CTD, $\mathrm{CF}_{ij}^{\mathrm{CTD}} = \frac{P_{ij}^*}{F_j^{\mathrm{CTD}}}, i=1:n, j=1:m$, and then find the theoretical quality option value, $\mathrm{ThQuOption} = \max_{i,j} F_j^{\mathrm{CTD}} * \left(\mathrm{CF}_i - \mathrm{CF}_{ij}^{\mathrm{CTD}}\right), i=1:n, j=1:m$;

Step 6: Find the two indexes, I and J, that maximize the theoretical quality option. If adjusted by this option, the future price is $F = \frac{P_{IJ}^*}{\mathrm{CF}_I}$.

4 | EMPIRICAL TEST

4.1 | Data

4.1.1 | Data of Treasury bond future

Our dataset covers the 10-year Treasury Bond Futures contract (T2403) expiring in March 2024 as the empirical research object. T2403 was listed on 12 June 2023. Figure 1 displays the 151 daily close prices from its listing date to 19 January 2024. Table 1 presents descriptive statistics.

The standard bond corresponding to Treasury bond futures contract T2403 is a 10-year nominal bond issued on 1 March 2024 with a coupon rate of 3%. The first and last deliverable dates are 1 March and 8 March 2024, respectively. During the 151 working days, the average futures price corresponding to the ￥100 face value is 101.74, the highest is 103.11, the lowest is 101.72, the median is 101.59, the skewness is 0.62 and the kurtosis is −0.67. Table 2 represents the 13 deliverable bonds corresponding to Treasury bond futures contract T2403 and their CFs.

Take the second bond (code 109646) as an example. The issue day is 19 November 2020 and the Maturity date is 19 November 2030, with the coupon rate of 3.27%. The Day Count is Act/365F, and the coupon dates are 19 November and 19 May each year. Then, corresponding to the first day of the maturity month (1 March 2024) of T2403, this bond has 12 cash flows left.

4.1.2 | Yield to maturity curve of the Treasury bond

We download all the term structures of the yield to maturity of Treasury bond from 12 June 2023 to 19 January 2024. Table 3 shows the curve for the first day.

4.2 | Pricing of the theoretical quality option and the Treasury bond future

Based on the above data, we conduct an empirical analysis on all data from 12 June 2023 to 19 January 2024 day

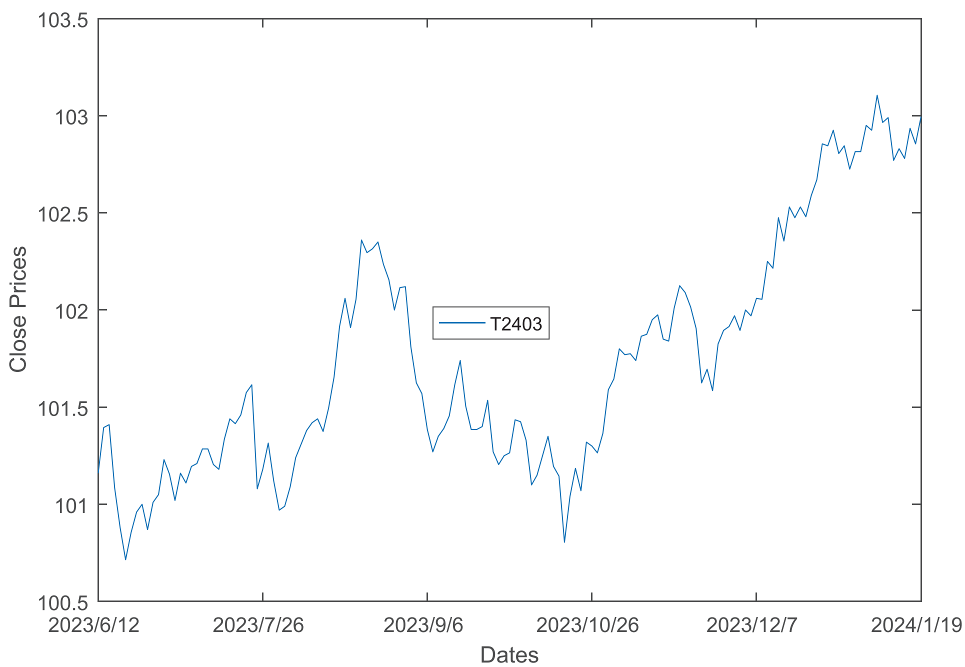

FIGURE 1 Daily close prices of T2403 from 2023/6/12 to 2024/1/19. *Source*: iFind system. [Colour figure can be viewed at wileyonlinelibrary.com]

TABLE 1 Descriptive statistics of the T2403 daily close prices.

No.	Mean	SD	Min	Max	MED	SKEW	KURT
151	101.74	0.61	101.72	103.11	101.59	0.62	−0.67

TABLE 2 T2403 deliverable bonds.

No.	Code	Issue date	Maturity	Frequency	Coupon (%)	Conversion factor
1	019722	15 Sep 2023	15 Sep 2030	1	2.6	0.9766
2	019646	19 Nov 2020	19 Nov 2030	2	3.27	1.0162
3	019731	25 Dec 2023	25 Dec 2030	1	2.54	0.9722
4	019657	27 May 2021	27 May 2031	2	3.02	1.0013
5	019665	18 Nov 2021	18 Nov 2031	2	2.89	0.9925
6	019668	17 Feb 2022	17 Feb 2032	2	2.75	0.9825
7	019675	15 May 2022	15 May 2032	2	2.76	0.9827
8	019682	15 Aug 2022	15 Aug 2032	2	2.69	0.9771
9	019690	15 Nov 2022	15 Nov 2032	2	2.8	0.9848
10	019697	25 Feb 2023	25 Feb 2033	2	2.88	0.9907
11	019705	25 May 2023	25 May 2033	2	2.67	0.9737
12	019721	25 Aug 2023	25 Aug 2033	2	2.52	0.9609
13	019729	25 Nov 2023	25 Nov 20233	2	2.67	0.9725

Note: Data source: iFind system.

TABLE 3 Key points of the yield to maturity curve on 12 June 2023.

Term (year)	Yield (%)	Term (year)	Yield (%)
0	1.35	5	2.4361
0.08	1.6097	7	2.6293
0.17	1.6488	10	2.6704
0.25	1.6652	15	2.8458
0.5	1.7884	20	2.9392
0.75	1.8211	30	3.0391
1	1.8663	40	3.1472
2	2.1696	50	3.1625
3	2.2421		

Note: Data source: www.chinabond.com.cn.

by day according to the steps mentioned in the third section. 12 June 2023 is an example of giving the results step by step, and then obtaining the final results for all of the 151 trading days according to the same method.

4.2.1 | Valuation of the bond

Table 3 offers the yield to maturity curve of Treasury bonds on 12 June 2023. Table 4 takes this day as the valuation date to present the rest of the cash flows and their corresponding dates of the T2403 standard bond.

Table 5 shows the values of the standard bond during the deliverable dates between 1 Mar 2024 and 8 Mar 2024, calculated using Equations (8) and (9).

TABLE 4 Cash flows after 12 June 2023 of the T2403 standard bond.

No.	1	2	3		18	19	20
Dates	1 Sep 2024	1 Mar 2025	1 Mar 2025	...	1 Mar 2033	1 Sep 2033	1 Mar 2034
Cash flow	1.5	1.5	1.5	...	1.5	1.5	101.5

TABLE 5 Values of the standard bond corresponding to six deliverable days.

Dates	1 Mar 2024	4 Mar 2024	5 Mar 2024	6 Mar 2024	7 Mar 2024	8 Mar 2024
Full	100.8591	100.8746	100.8797	100.8849	100.8901	100.8953
AI	0	0.0245	0.0326	0.0408	0.0489	0.0571
Clean	100.8591	100.8501	100.8471	100.8441	100.8412	100.8382

TABLE 6 Clean prices on 12 June 2023.

	Dates					
Bonds	1 Mar 2024	4 Mar 2024	5 Mar 2024	6 Mar 2024	7 Mar 2024	8 Mar 2024
1	NaN	NaN	NaN	NaN	NaN	NaN
2	103.0884	103.0774	103.0737	103.0701	103.0664	103.0628
3	NaN	NaN	NaN	NaN	NaN	NaN
4	101.5984	101.5892	101.5861	101.5831	101.5801	101.5771
5	100.8186	100.8103	100.8075	100.8048	100.8021	100.7994
6	99.8858	99.8784	99.8760	99.8736	99.8712	99.8688
7	99.9967	99.9893	99.9869	99.9845	99.9821	99.9797
8	99.5188	99.5119	99.5096	99.5073	99.5051	99.5028
9	100.3894	100.3818	100.3793	100.3768	100.3743	100.3718
10	101.0606	101.0523	101.0496	101.0469	101.0442	101.0415
11	99.3923	99.3856	99.3834	99.3812	99.3791	99.3769
12	NaN	NaN	NaN	NaN	NaN	NaN
13	NaN	NaN	NaN	NaN	NaN	NaN

TABLE 7 Quotations of the Treasury bond future on the CTD.

Dates	1 Mar 2024	4 Mar 2024	5 Mar 2024	6 Mar 2024	7 Mar 2024	8 Mar 2024
F_j^{CTD}	102.0770	102.0701	102.0678	102.0656	102.0633	102.0611

Table 6 displays the clean prices of each deliverable bond on different deliverable dates according to the same method. This table has no value for the 1st, 3rd, 12th, 13th bonds since they are not listed on this day.

4.2.2 | CTD bond and the cost of carry future price F^{CTD}

Combining the CFs according to Equation (3), the 11th bond (code 019705) is the CTD bond, delivered on the 6th delivery date. Then, $F_j^{\mathrm{CTD}} = \frac{P^*_{\mathrm{CTD},j}}{\mathrm{CF_{CTD}}} = \frac{P^*_{11,j}}{0.9737}$ calculates the quotation of Treasury bond futures corresponding to the CTD (i.e., the theoretical quotation of the cost of carry model), as shown in Table 7.

4.2.3 | Theoretical CF on the CTD bond

Table 8 represents the theoretical CF based on the CTD bond of different bonds corresponding to different delivery dates, calculated by $\mathrm{CF}_{ij}^{\mathrm{CTD}} = \frac{P^*_{ij}}{F_j^{\mathrm{CTD}}}$. The theoretical CFs of the 11th bond are consistent with their corresponding actual CFs.

4.2.4 | Theoretical quality option value and the final future price

Maximize the formula $\max\limits_{i,j} F_j^{\mathrm{CTD}} * \left(\mathrm{CF}_i - \mathrm{CF}_{ij}^{\mathrm{CTD}}\right)$ gives the bond and delivery dates. The calculation shows that from the view on 12 June 2023, the short side of futures

TABLE 8 Theoretical conversion factor on CTD on 12 June 2023.

	Dates					
Bonds	1 Mar 2024	4 Mar 2024	5 Mar 2024	6 Mar 2024	7 Mar 2024	8 Mar 2024
1	NaN	NaN	NaN	NaN	NaN	NaN
2	1.0221	1.0221	1.0221	1.0221	1.0221	1.0221
3	NaN	NaN	NaN	NaN	NaN	NaN
4	1.0073	1.0073	1.0073	1.0073	1.0073	1.0073
5	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996
6	0.9903	0.9904	0.9904	0.9904	0.9904	0.9904
7	0.9914	0.9915	0.9915	0.9915	0.9915	0.9915
8	0.9867	0.9867	0.9867	0.9867	0.9868	0.9868
9	0.9953	0.9954	0.9954	0.9954	0.9954	0.9954
10	1.0020	1.0020	1.0020	1.0020	1.0020	1.0020
11	0.9855	0.9855	0.9855	0.9855	0.9855	0.9855
12	NaN	NaN	NaN	NaN	NaN	NaN
13	NaN	NaN	NaN	NaN	NaN	NaN

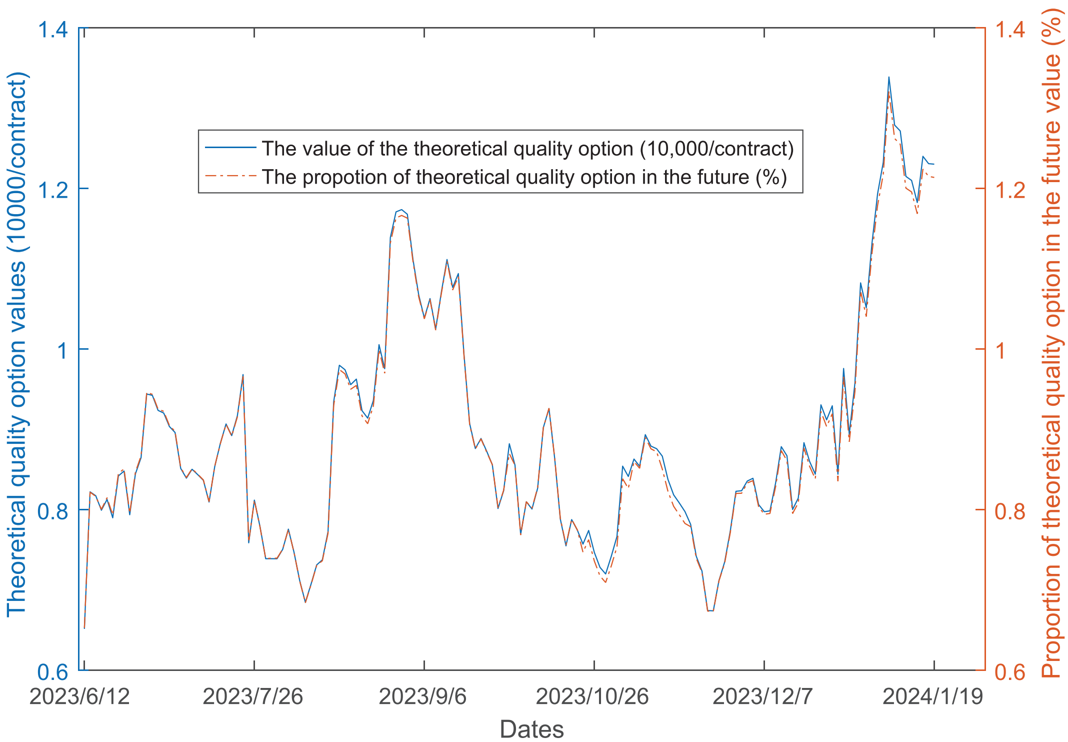

FIGURE 2 Values of theoretical quality option and the proportion. [Colour figure can be viewed at wileyonlinelibrary.com]

select the second bond (code 019646) to deliver on the last delivery date (the 6th delivery date) to make the option reach the maximum value of 6517 Yuan which is the value of the theoretical quality option. Then, $F = \frac{P^*_{2,6}}{CF_2} = \frac{103.0628}{1.0162} = 101.4198$ calculates the future price, and the actual close price is 101.16.

This method also calculates the theoretical quality option values and futures prices for all the 151 trading days from 12 June 2023 to 19 January 2024. Figure 2 shows the values of theoretical quality options and their proportion to the total value of futures (calculated by futures price $\times$ CF). In the figure, the solid line represents the values of theoretical quality option, the ordinate refers to the left axis, the dotted line represents the proportion of theoretical option value in the total value of futures, and the ordinate refers to the right axis. Based on the figure, the maximum value of the theoretical quality option is about 13,387 Yuan per contract, accounting for about 1.3% of the total value of futures. The average value of theoretical quality options in these 151 days is 8925 Yuan, accounting for an average of 0.88%, which is not small.

Figure 3 shows the comparison among the theoretical futures prices of the cost of carry model (the top dotted line), the futures price excluding the theoretical quality option (the solid line), and the actual closing price of Treasury bond futures T2403 (the bottom dotted line). With regard to the figure, the futures price under the cost of carry model has always been above the other two lines, while our estimated Treasury bond futures price is between the actual closing price and the cost of carry model futures price. The trends of the three lines are basically consistent, proving the model effectiveness. In addition, the estimated price of Treasury bond futures in the figure almost all are above the actual price, indicating that the embedded rolling timing option and month end option of Treasury bond futures also need consideration, and the model needs expansion.

Table 9 lists the descriptive statistics of the deviations between the futures price of the two models and the closing prices of the Treasury bond futures. The average deviation of the cost of carry model is 10,661 Yuan (with the proportion of 1.05%), the maximum and minimum are 15,142 and 6440 Yuan, respectively, and the standard deviation is 2036 Yuan. The maximum and minimum deviations of our model are 6496 and 16.26 Yuan, respectively. The average is about 1867 Yuan (with a proportion of 0.22%), and the standard deviation is 1454 Yuan. The relative deviation decreases by about half compared with the cost of carry model. Therefore, the embedded quality option pricing method in this article is more accurate and has a particular and practical value.

According to the statistical analysis on optimal execution dates during the 151 days, the number of days for optimal delivery on the last, first, and other delivery days are 151, 0, and 0, respectively. This result is consistent

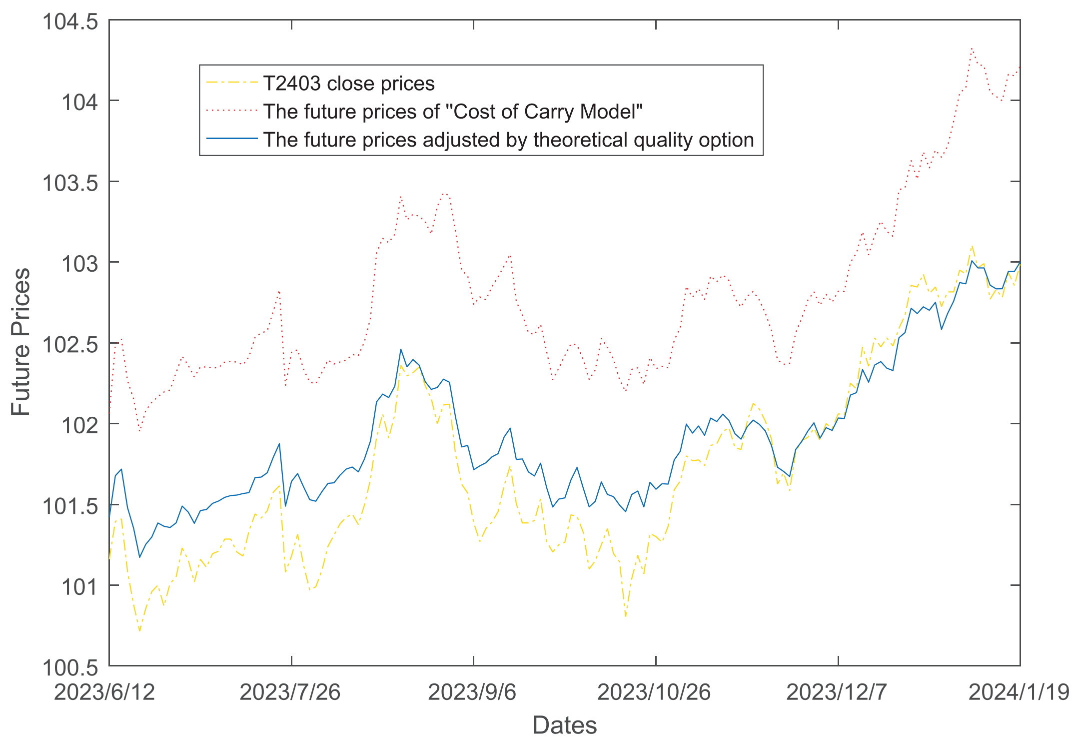

FIGURE 3 Comparison chart of three future prices. [Colour figure can be viewed at wileyonlinelibrary.com]

TABLE 9 Absolute deviations of the two models from actual price (Yuan/contract).

Model	Max	Min	Average (proportion)	Std
Cost of carry model	15,142	6440	10,661 (1.05%)	2036
Our model	6496	16.26	1867 (0.22%)	1454

with Michèle and Ramzi (2018). They analysed all CBOT Treasury bond futures delivery data from 1985 to 2016, concluded that most deliveries are postponed to the final deliverable day, and implemented in advance only under some special circumstances.

4.3 | Conclusion and further research

Based on the definition deviation of CF and the bond valuation method by yield to maturity curve, this article prices the theoretical quality options caused by this deviation, and constructs a new pricing model of Treasury bond futures. This model is relatively simple and easy to operate due to the lack of complex derivation and difficult concepts.

The empirical results show that the theoretical quality option value accounts for a relatively large proportion of the total value of futures, which is consistent with the conclusions of some theoretical literature. After considering the theoretical quality options, the futures pricing results in this article are closer to the actual futures closing price, compared with the cost of carry model, which disregards any embedded options. This result implies that the pricing model in this article is more accurate and has a particular and practical value.

To price a Treasury bond future using this model, this research suggests practitioners to follow a process with the following steps (outlined in Section 4). The first step gathers relevant data on the deliverable bonds and standard bonds corresponding to the Treasury bond future, including coupons, coupon dates, maturities, and CFs. The second step obtains the term structure of Treasury bond yields on the valuation date. The third step calculates the forward clean prices for different deliverable dates. The fourth step identifies the CTD bond using Equation (3) and calculates the cost of carry model quotation. The fifth step calculates the adjusted theoretical CFs using Equation (4). The sixth step determines the theoretical quality option value and identify the final deliverable bond considering quality options. The final step calculates the future price.

Since this article only considers the embedded theoretical quality options, and excludes the measurement of the other two timing options, the calculated futures price is basically above the actual closing price of Treasury bond futures. Future studies can use this article to introduce these two parts, combine them with theoretical quality options to price Treasury bond futures, and put forward a more comprehensive pricing model. In addition, this article ignores the time value of cash flows during the deliverable dates when calculating the theoretical CF, which is another direction for improving this model.

Finally, the statistical analysis of the optimal delivery dates shows that investor mostly delivers on the last deliverable date. This phenomenon is a feasible entry point for the future modelling and analysis of timing options.

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DATA AVAILABILITY STATEMENT

The data that supports the findings of this study are available in the supplementary material of this article.

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ENDNOTES

[1] In order to simplify the delivery and calculation process, China Financial Futures Exchange has given a simplified definition as international practice.

[2] In the rolling delivery stage, the time for determining the settlement price is consistent with the deadline for submission of delivery intention in China's Treasury bond futures market. In the US, however, the settlement price of the day is determined at 2 p.m. and the short party of the futures is allowed to submit the delivery intention at any time before 8 p.m. in that day.

[3] Clean price = Dirty price − Accrued interest. The dirty price represents the total value of a coupon bond, and the accrued interest is calculated by a fixed method based on the bond's day-count convention. For example, in the last part of Equation (1), the first and second half represent the dirty price and accrued interest, respectively. Similar examples can be found in Equation (2), where P^*_{ij} and F^*_j indicate clean prices.

[4] This paper utilizes the market-available yield to maturity curve for discounting purposes.

[5] The official website of China Central Depository Clearing Co., Ltd has no details about the definition of d_i. By comparation, we choose the definition which make the interpolate results closer to the published yields, that is $d_i = \frac{y_i - y_{i-1}}{t_i - t_{i-1}}$ and $d_{i+1} = \frac{y_{i+1} - y_i}{t_{i+1} - t_i}$.

REFERENCES

Balbas, A., & Reichardt, S. (2010). On the future contract quality option: A new look. *Applied Financial Economics*, *20*(15), 1217–1229.

Björk, T. (1997). Interest rate theory. In W. J. Runggaldier (Ed.), *Financial mathematics*. Lecture Notes in Mathematics (Vol. 1656, pp. 53–122). Springer. https://doi.org/10.1007/BFb0091999

Carr, P. (1988). Valuing bond futures and the quality option. Working paper, University of California at Los Angeles.

Carr, P., & Chen, R. (1996). Valuing bond futures and the quality option. Working paper, Rutgers University.

Chan, K. C., Karoyli, G. A., Longstaff, F. A., & Sanders, A. B. (1992). An empirical comparison of alternative models of the short term interest rate. *Journal of Finance*, *47*, 1209–1227.
Chen, R., & Ge, J. (2017). Pricing the China's Treasury bond futures and the implied quality options. *Journal of Applied Statistics and Management*, *36*(2), 361–380.
Cornell, B., & French, K. (1983a). Taxes and the pricing of stock index futures. *Journal of Finance*, *38*(3), 675–694.
Cornell, B., & French, K. (1983b). The pricing of stock index futures. *Journal of Future Markets*, *3*, 1–14.
Cox, J. C., Ingersoll, J. E., & Ross, S. A. (1985). A theory of the term structure of interest rates. *Econometrica*, *53*, 129–151.
Gay, G. D., & Manaster, S. (1984). The quality option implicit in future contracts. *Journal of Financial Economics*, *13*, 353–370.
Grieves, R., & Marcus, A. J. (2005). Delivery options and Treasury bond futures Hedge ratios. *Journal of Derivatives*, *13*(2), 70–76.
Grieves, R., Marcus, A. J., & Woodhams, A. (2010). Delivery options and convexity in Treasury bond and futures. *Review of Financial Economics*, *19*, 1–7.
Gürkaynak, R. S., Sack, B., & Wright, J. (2007). The US Treasury yield curve: 1961 to the present. *Journal of Monetary Economics*, *54*, 2291–2304.
Heath, D., Jarrow, R., & Morton, A. (1992). Bond pricing and the term structure of interest rates: A new methodology for contingent claims valuation. *Econometrica*, *60*(1), 77–105.
Hedge, S. P. (1988). An empirical analysis of implicit delivery options in the Treasury bond futures contract. *Journal of Banking & Finance*, *12*, 469–492.
Hedge, S. P. (1990). An ex-post valuation of the quality option implicit in the Treasury bond futures contract. *Journal of Banking & Finance*, *14*, 741–760.
Hemler, M. (1990). The quality delivery option in Treasury bond futures contracts. *The Journal of Finance.*, *45*(5), 1565–1586.
Kane, A., & Marcus, A. J. (1984). Conversion factor risk and hedging in the Treasury bond futures market. *The Journal of Futures Market*, *4*(1), 55–64.
Kane, A., & Marcus, A. J. (1986a). The quality option in Treasury bond futures market: An empirical assessment. *Journal of Futures Markets*, *6*(2), 231–248.
Kane, A., & Marcus, A. J. (1986b). Valuation and optimal exercise of the wild card option in the Treasury bond futures market. *The Journal of Finance*, *41*(1), 195–207.
Margrabe, W. (1978). The value of an option to exchange one asset for another. *The Journal of Finance*, *33*, 177–186.
Michèle, B., & Ramzi, B.-A. (2018). Time is money: An empirical investigation of delivery behavior in the U.S. T-bond futures market. *Journal of Futures Markets*, *38*(1), 22–37.
Nelson, C. R., & Siegel, A. F. (1987). Parsimonious modeling of yield curves. *Journal of Business*, *60*, 473–489.
Nunes, J. P. V., & De Oliveira, L. A. F. (2007). Multifactor and analytical valuation of Treasury bond futures with an embedded quality option. *The Journal of Futures Markets*, *27*(3), 275–303.
Oviedo, R. (2006). Improving the design of Treasury bond future contracts. *The Journal of Business*, *79*(3), 1293–1315.
Peter, H. L. (1982). Large sample properties of generalized method of moments estimators. *Econometrica*, *50*, 1029–1054.
Ramzi, B.-a., Haterm, B.-A., & Michèle, B. (2009). An analysis of the true notional bond system applied to the CBOT T-bond futures. *Journal of Banking & Finance*, *33*, 534–545.
Ramzi, B.-A., & Michèle, B. (2017). History is repeating itself: Get ready for a long dry spell. *Financial Analysts Journal*, *73*(3), 106–130.
Rendleman, R. J. (2004). Delivery options in the pricing and hedging of Treasury bond and note futures. *Journal of Fixed Income*, *14*, 20–31.
Ritchken, P., & Sankarasubramanian, L. (1992). Pricing the quality option in Treasury bond futures. *Mathematical Finance*, *2*(3), 197–214.
Stulz, R. M. (1982). Options on the minimum or maximum of two risky asset. *Journal of Financial Economics*, *10*(2), 161–185.
Sun, C. (2014). An empirical study on option pricing of China's Treasury bond futures delivery. *Contemporary Economic*, *18*, 130–131.
Svensson, L. E. O. (1994). Estimating and interpreting forward rates: Sweden 1992–4. National Bureau of Economic Research Working Paper, #4871.
Vasicek, O. (1977). An equilibrium Characterizatioin of the term structure. *Journal of Financial Economics*, *5*, 177–188.
Wang Su-Sheng, Y., & Yong-Rui, K. Y.-B. (2017). Research on the pricing of China's Treasury bond futures and the cheapest deliverable bonds. *Journal of North China Electric Power University (Social Science Edition)*, *2*, 76–81.
Xiong, Y.-J. (2019). Research on the pricing of Treasury bond futures under dynamic interest rate model. Master's Thesis of Yunnan University.
Yong, X. (2018). *The choice of financing interest rate in the pricing of Treasury bond future*. Bond.
Zeng, G.-M. (2015). Research on the pricing of China's Treasury bond futures—Based on the correction method of delivery options. *Securities Market Herald*, *10*, 36–47.
Zhang, C.-K. (2013). Classification and measurement of Treasury bond futures delivery options. *China Bond*, *8*, 51–53.

SUPPORTING INFORMATION

Additional supporting information can be found online in the Supporting Information section at the end of this article.

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APPENDIX A: DIFFERENT METHOD TO CALCULATE THE CLEAN PRICE OF BONDS IN MATURITY

Equations (A1) and (A2) represent the stochastic models developed by Vasicek (1977) and Cox-Ingersoll-Ross (CIR) (1985), respectively, to describe the evolution of interest rate movements over time.

1. Vasicek

$$dr_t = \kappa(\theta - r_t)dt + \sigma dW_t. \tag{A1}$$

2. CIR

$$dr_t = \kappa(\theta - r_t)dt + \sigma\sqrt{r_t}dW_t. \tag{A2}$$

The Vasicek model assumes a mean-reverting process for interest rates, suggesting that rates tend to revert to a long-term mean or equilibrium rate over time. Its limitation is the resulting negative interest rates, which are inconsistent with some economic contexts. To address this issue, the CIR model, extends the Vasicek model by incorporating a square root process that prevents negative rates. In this way, it appropriately models interest rates in environments where negative rates are infeasible. Both models are affine, that is, their solutions have a simple closed-form expression, making them computationally efficient.

This study initially opted to calculate the clean prices of deliverable bonds and standard bonds using the Vasicek model. According to this process, Equation (A3) represents the instantaneous interest rate at future time t.

$$r(t) = e^{-\kappa(t-s)}r(0) + \theta\left(1 - e^{-\kappa(t-s)}\right) + \sigma\int_0^t e^{-\kappa(t-u)}dW(u). \tag{A3}$$

Equations (A4) and (A5) show the expectation and variance, respectively.

$$E\{r(t)|\mathcal{F}_0\} = r(0)e^{-\kappa t} + \theta(1 - e^{-\kappa t}), \tag{A4}$$

$$\mathrm{Var}\{r(t)|\mathcal{F}_0\} = \frac{\sigma^2}{2\kappa}\left(1 - e^{-2\kappa t}\right). \tag{A5}$$

Chan et al. (1992) provided a comprehensive account of parameter calibration for interest rate models. This calibration considers that $\varepsilon_{t+1} = r_{t+1} - r_t - \kappa\theta\Delta t + \kappa r_t\Delta t$, with the distribution $\varepsilon_{t+1} \sim \emptyset(0, \sigma^2\Delta t)$, where ε_{t+1} and r_t are independent. Given these assumptions, Peter (1982) suggests to employ the Generalized Method of Moments (GMM) to estimate the parameters using historical interest rate yield curve.

According to Björk (1997), the principle of no arbitrage dictates that the value of zero-coupon bonds with any maturity time has an affine analytical solution under Vasicek model (Similar, yet more complex, results can be found under the CIR model):

$$p(t,T) = A_t(T) \times e^{-B_t(T)r(t)}, \tag{A6}$$

where

$$B_t(T) = \int_t^T e^{-as}ds = \frac{1}{\kappa}\left\{1 - e^{-\kappa(T-t)}\right\},$$

$$A_t(T) = \exp\left\{\left(\theta - \frac{\sigma^2}{2\kappa^2}\right)[B_t(T) - T + t] - \frac{\sigma^2}{4\kappa}B_t^2(T)\right\}.$$

In other words, if the instantaneous interest rate $r(t)$ at time t is known, the value of any discounted bond that matures at time T from time t is also determined.

It is well-established that the theoretical CF relies on the discount of residual cash flows on the delivery date. Moreover, the clean price of deliverable and standard bonds in Equation (2) can be expressed as a combination of a series of discounted bonds. Then, we have:

$$\begin{aligned} P_{ij}^* = {} & C^i \times p\left(t, t_1^{ij}\right) + C^i \times p\left(t, t_2^{ij}\right) + \ldots \\ & + \left(C^i + 1\right) \times p\left(t, t_{n_{ij}}^{ij}\right) - AI_{ij}\; i = 1:n, j = 1:m, \end{aligned} \tag{A7}$$

$$\begin{aligned} F_j^* = {} & C^{`} \times p\left(t, t_1^j\right) + C^{`} \times p\left(t, t_2^j\right) + \ldots \\ & + \left(C^{`} + 1\right) \times p\left(t, t_{n_j}^j\right) - AI_j\, j = 1:m. \end{aligned} \tag{A8}$$

Equation (A8) shows that all $p(t,T)$ solely depend on $r(t)$. However, the actual value of $r(t)$ on the day of pricing Treasury bond futures is unknown. Consequently, we utilize the expected value of $r(t)$ instead. By substituting Equation (A4) into Equation (A6), we calculate $p(t,T)$. Subsequently, Equations (A7) and (A8) determine the clean prices of deliverable and standard bonds, respectively.

After calculating the future clean prices, the subsequent steps align with this article.